

# On Unitary Time Evolution in Gowdy $T^3$ Cosmologies

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## Abstract

A non-perturbative canonical quantization of Gowdy  $T^3$  polarized models carried out recently is considered. This approach profits from the equivalence between the symmetry reduced model and 2+1 gravity coupled to a massless real scalar field. The system is partially gauge fixed and a choice of internal time is performed, for which the true degrees of freedom of the model reduce to a massless free scalar field propagating on a 2-dimensional expanding torus. It is shown that the symplectic transformation that determines the classical dynamics cannot be unitarily implemented on the corresponding Hilbert space of quantum states. The implications of this result for both quantization of fields on curved manifolds and physically relevant questions regarding the initial singularity are discussed.

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## I. INTRODUCTION

In the search for a quantum theory of gravity within the canonical approach, it has been historically useful to consider symmetry reduced models. The most studied examples are homogeneous models, where the infinite dimensional system is reduced to a model with a finite number of degrees of freedom. These are known as mini-superspace models [1]. Another class of symmetry reduced models where the resulting system is still a field theory with an infinite number of degrees of freedom are known as midi-superspace models (for a recent review see [2]). Recently, within this class, the models that have received special attention are the Einstein-Rosen waves and Gowdy cosmological models [3–5]. One interesting feature of this type of models is that due to their spacetime symmetries, the classical dynamics turns out to be derivable from an equivalent complete integrable system, but they still possess an infinite number of degrees of freedom so that their quantization would lead to a true quantum field theory (in contrast to mini-superspace models with a finite number of degrees of freedom). In this work, we will consider the canonical quantization of the polarized Gowdy  $T^3$  cosmological model [6]. This is the simplest inhomogeneous, empty, spatially closed cosmological model. It was extensively studied in the 70's [7,8], and subsequently re-examined by several authors [9,10]. Of particular relevance is the recent work by Pierri, where definite progress was achieved in defining a rigorous quantization of the model [4]. In this case, the quantization is based on the fact that the corresponding gravitational field can be equivalently treated as  $2 + 1$  gravity coupled to a massless scalar field, defined on a  $2 + 1$ -dimensional manifold with topology  $T^2 \times R$ .

One important aspect in the study of quantum cosmological models is dynamical evolution; that is the dynamics in which a physical quantum state evolves from an initial Cauchy surface to a final one. Recall that in general relativity there are no preferred foliations in spacetime and the dynamical evolution should consider all possible spacelike foliations in order to be in agreement with the requirement of general covariance. Furthermore, in the case in which the Cauchy surfaces are compact, dynamical evolution is pure gauge, so any interpretation of time evolution is normally through a deparametrization procedure which, in the Hamiltonian language, is normally achieved via “time dependent gauge fixing”. Thus, the dynamics to be considered in quantum cosmological models concerns the evolution of quantum states between Cauchy surfaces, defined by the particular gauge choice. Different choices of time parameters may lead to inequivalent quantizations. This fact is true even in the simple mechanical mini-superspace models. In the particular model we are interested, the system is partially gauge fixed at the classical level, and in particular a time function  $T$  is chosen and interpreted as the time which defines “evolution”. The surfaces of constant  $T$  are Cauchy surfaces of a fiducial flat background, so the model get reduced to a quantum scalar field on a flat background, equipped with a foliation of preferred surfaces which define “time evolution” in the corresponding quantum gravitational system.

At the classical level, this dynamical evolution of the field can be represented as a canonical transformation that acts on points of the corresponding phase space. The question is whether, at the quantum level, this canonical transformation can be implemented on the space of quantum states of the field by means of a unitary operator. This is a rather delicate problem that has been analyzed in detail only recently and for a few special cases [11–14], all of them concerning free scalar fields on flat or stationary spacetimes. Fortunately, the

quantum polarized Gowdy  $T^3$  cosmological models belong to this class, and so we will be able to investigate the question about the unitary implementability of these models within this approach. This is the main goal of the present work.

We will show that for the particular quantization performed in [4,5], time evolution is not implementable as a unitary evolution. Given that, at the classical level time evolution is pure gauge, and the particular choice of time is an ad-hock procedure to regain dynamics from a purely frozen formalism, one might argue that unitary implementability of this fictitious time evolution is not necessary for the consistency of the formalism. However, as we will argue, the implementability is needed in order to ask physically meaningful questions regarding, say, the initial singularity. That is, questions such as whether the initial singularity is smeared by quantum effects should have a definite answer within a consistent quantization. Thus, we shall conclude that we can not extract any physics out of these models as presently constructed.

This paper is organized as follows. In Sec. II we review the quantization of polarized Gowdy  $T^3$  cosmological models as performed by Pierri [4,5]. We show that the two different sets of creation and annihilation operators proposed in this quantization are related by means of a unitary Bogoliubov transformation. In Sec. III A we explicitly calculate the canonical (symplectic) transformation that represents the classical evolution of the system. In Sec. III B we prove that this canonical transformation is not unitarily implementable on the corresponding Fock space. We end with a discussion and some conclusions in Sec. IV.

## II. CANONICAL QUANTIZATION

The polarized Gowdy  $T^3$  models are globally hyperbolic four-dimensional vacuum spacetimes, with two commuting hypersurface orthogonal spacelike Killing fields and compact spacelike hypersurfaces homeomorphic to a three-torus. Because this system can be equivalent treated as 2+1 gravity (minimally) coupled to an axi-symmetric massless scalar field, let us begin by considering the action

$$S[{}^{(3)}g, \psi] = \frac{1}{2\pi} \int_{{}^{(3)}M} d^3x \sqrt{-{}^{(3)}g} ({}^{(3)}R - {}^{(3)}g^{ab} \nabla_a \psi \nabla_b \psi) \quad (1)$$

where  ${}^{(3)}R$  is the Ricci scalar of the 3-d spacetime  $({}^{(3)}M, {}^{(3)}g_{ab})$ ,  ${}^{(3)}M$  is a 3-d manifold with topology  $T^2 \times \mathbf{R}$  and spacetime metric  ${}^{(3)}g_{ab} = h_{ab} + \tau^2 \nabla_a \sigma \nabla_b \sigma$ . The Killing field  $\sigma^a$  is hypersurface orthogonal and the field  $h_{ab}$  is a metric of signature  $(-, +)$  on the 2-manifold orthogonal to  $\sigma^a$ ;  $\tau$  is the norm of  $\sigma^a$  and  $\sigma$  is an angular coordinate with range  $0 \leq \sigma < 2\pi$  such that  $\sigma^a \nabla_a \sigma = 1$ .

Introducing a generic slicing by compact spacelike hypersurfaces labeled by  $t = \text{const}$  the 2-metric can be written as  $h_{ab} = (-N^2 + N^\theta N_\theta) \nabla_a t \nabla_b t + 2N_\theta \nabla_{(a} t \nabla_{b)} \theta + e^\gamma \nabla_a \theta \nabla_b \theta$ , where the lapse,  $N$ , the shift,  $N^\theta$ , and  $\gamma$  are functions of  $\theta$  and  $t$ . The angular coordinate  $\theta \in [0, 2\pi)$  is such that  $\hat{\theta}^a \nabla_a \theta = 1$ , where  $\hat{\theta}^a$  is the unit vector field within each slice orthogonal to  $\sigma^a$ . Thus the system consists of five functions  $(N, N^\theta, \gamma, \tau, \psi)$  of  $t$  and  $\theta$  which are periodic in  $\theta$ . The function  $\psi$  represents the zero rest mass scalar field.

Substituting the expression for  ${}^{(3)}g_{ab}$  in the action (1) we pass to the Hamiltonian formulation

$$S = \int dt \left( \oint (p_\gamma \dot{\gamma} + p_\tau \dot{\tau} + p_\psi \dot{\psi}) \right) - H[N, N^\theta] \quad (2)$$

where the Hamiltonian  $H$  is given by  $H[N, N^\theta] = \oint (NC + N^\theta C_\theta)$  (here, the symbol  $\oint$  denotes integration over  $\theta \in S^1$ ) and the first class constraints  $C$  and  $C^\theta$  are

$$C = e^{-\gamma/2} \left[ 2\tau'' - \gamma'\tau' - p_\gamma p_\tau + \tau \left( \frac{p_\psi^2}{4\tau^2} + \psi'^2 \right) \right] \quad (3)$$

$$C^\theta = e^{-\gamma} (\gamma' p_\gamma - 2p'_\gamma + \tau' p_\tau + p_\psi \psi') \quad (4)$$

The lapse and shift are not dynamical variables, thus the phase space  $\Gamma$  consists of three canonically-conjugate pairs of periodic functions of  $\theta$ ,  $(\gamma, p_\gamma; \tau, p_\tau; \psi, p_\psi)$  on a 2-d manifold  $\Sigma$  with topology  $T^2$ .

Because the Hamiltonian vanishes on the constraint surface there is no distinction between gauge and dynamics and therefore it is necessary to introduce a “deparametrization” procedure to discuss dynamics. From the infinite set of vector fields generated by the Hamiltonian constraints we select one to represent evolution and gauge fix the others. For gauge fixing let us demand

$$p_\gamma + p = 0 \quad \text{and} \quad \tau(\theta)' = 0 \quad (5)$$

where  $p$  is a spatial constant that has zero Poisson bracket with all the constraints and hence it can not be removed by gauge fixing. The second condition will allow us to regard  $\tau(\theta)$  as the time parameter.

The consistency of the formalism requires that the Poisson brackets  $\{\tau(\theta)', H[N, N^\theta]\}$  and  $\{p_\gamma + p, H[N, N^\theta]\}$  vanish. This can be achieved if the freely specifiable  $N$  and  $N^\theta$  are chosen as,

$$N = \frac{e^{\gamma/2}}{p} \quad \text{and} \quad N^\theta = 0 \quad (6)$$

therefore the coordinate condition (5) is acceptable. Indeed, for the special choice (6) we have that  $\dot{\tau}(\theta) = 1$  and  $\dot{p}_\gamma(\theta) = -\dot{p} = 0$  and hence  $p$  becomes a true constant that can be associated with a time scaling parameter in the original 3+1-spacetime. Thus, apart from a global degree of freedom, the true degrees of freedom will all reside in the field  $\psi$ . Solving the set of second class constraints  $(C, C^\theta, p_\gamma + p, \tau')$  the result is

$$p_\tau = -\frac{t}{p} \left( \frac{p_\psi^2}{4t^2} + \psi'^2 \right) \quad (7)$$

$$\gamma(\theta) = \frac{1}{p} \int_0^\theta d\theta_1 p_\psi \psi' + \gamma(0) \quad (8)$$

Since  $\gamma$  is a smooth function of  $\theta$ , it must admit a Fourier series of the form  $\gamma = q + \sum_{n \neq 0} \frac{e^{in\theta}}{\sqrt{2\pi}} \gamma_n$ , then  $\sum_{n \neq 0} \left( \frac{e^{in\theta} - 1}{\sqrt{2\pi}} \right) \gamma_n = \frac{1}{p} \int_0^\theta d\theta_1 p_\psi \psi'$  and we can solve for all modes but the zero mode. i.e., we can solve (8) for  $\bar{\gamma} := \sum_{n \neq 0} \frac{e^{in\theta}}{\sqrt{2\pi}} \gamma_n$  and we are left with the global degree of

freedom  $q = \frac{1}{2\pi} \oint \gamma$  unsolved. This is consistent with the fact that a constant shift vector  $N^\theta = \text{const}$  would also be acceptable for preserving the conditions (5). Substituting (8) in (6) we obtain that  $N = N[q, p, \psi, p_\psi]$ . The spacetime metric is now completely determined by  $q, p, \psi$  and  $p_\psi$

$$^{(3)}g_{ab} = e^{q+\bar{\gamma}} \left( -\frac{1}{p^2} \nabla_a t \nabla_b t + \nabla_a \theta \nabla_b \theta \right) + t^2 \nabla_a \sigma \nabla_b \sigma \quad (9)$$

The phase space variables are periodic functions of  $\theta$ , therefore  $\gamma(2\pi) - \gamma(0) = 0$  defines via Eq.(8) the global constraint

$$P_\theta := \oint p_\psi \psi' = 0 \quad (10)$$

The non-degenerate symplectic structure on the reduced phase space  $\Gamma_r = \Gamma_g \oplus \bar{\Gamma}$ , where  $\Gamma_g$  is coordinatized by the pair  $(q, p)$  and  $\bar{\Gamma}$  by  $(\psi, p_\psi)$ , is the pull-back of the natural symplectic structure defined on  $\Gamma$ . Thus  $\{q, p\} = 1$  and  $\{\psi(\theta_1), p_\psi(\theta_2)\} = \delta(\theta_1, \theta_2)$  on  $\Gamma_r$ .

Substituting (5) and (7) in (2) we obtain the reduced action

$$S = \int dt \left( p\dot{q} + \oint \left[ p_\psi \dot{\psi} - \frac{t}{p} \left( \frac{p_\psi^2}{4t^2} + \psi'^2 \right) \right] \right) \quad (11)$$

and the reduced Hamiltonian

$$H = \oint \frac{t}{p} \left[ \frac{p_\psi^2}{4t^2} + \psi'^2 \right] \quad (12)$$

Varying the action (11) with respect to  $\psi$  and  $p_\psi$ , the field equations are  $\frac{\partial \psi}{\partial T} = \frac{p_\psi}{2Tp}$  and  $\frac{\partial p_\psi}{\partial T} = 2Tp\psi''$ , which is equivalent to the Klein-Gordon equation for the scalar field  $\psi$  propagating on a fictitious flat background  $^{(\epsilon)}g_{ab} = -\nabla_a T \nabla_b T + \nabla_a \theta \nabla_b \theta + T^2 \nabla_a \sigma \nabla_b \sigma$ , with the further restriction that the field  $\psi$  does not depend on  $\sigma$ . Here the constant rescaling  $T := t/p$  has been considered, in order to simplify the resulting dynamical equation for  $\psi$ . Hence the phase space  $\bar{\Gamma}$ , coordinatized by  $\varphi := \psi(\theta)$  and  $\pi := p_\psi(\theta)$ , corresponds to the symplectic vector space  $(V, \Omega_V)$  of smooth real solutions to the Klein-Gordon equation  $^{(\epsilon)}g^{ab} \nabla_a \nabla_b \psi = 0$ , where the symplectic structure  $\Omega_V$  is given by

$$\Omega_V(\psi_1, \psi_2) = \oint T(\psi_2 \partial_T \psi_1 - \psi_1 \partial_T \psi_2) \quad (13)$$

Note that the choice of internal time  $T : \Gamma \mapsto \mathbf{R}$ , depends on the point of phase space, in particular, on the global degree of freedom  $p$ , making it a  $q$ -number. Recall that in this case, one has deparametrized the system, that is, one has defined a fictitious time evolution with respect to the number  $t$ . Strictly speaking, from the canonical viewpoint one should choose a particular value  $t_0$  of the time parameter in order to fix once and for all a single Cauchy surface  $\Sigma_0$ . This would be the “frozen formalism” description, where only the true degrees of freedom are left and the notion of dynamics has been lost. However, for the purposes of quantization, it is convenient to exploit this deparametrization since this allows to complete the quantization in a rigorous fashion. However, in the quantum theory the

role of  $T$  is very different, namely the function  $T$  does not have an a-priori meaning as a spacetime time parameter. At best, one might expect that if one chooses suitable semi-classical states, a classical notion of time might arise, which could be then compared to the function  $T$ . We shall come back to the issue of “frozen formalism vs. fake dynamics” in the discussion section.

Thus, the problem of quantization of the true degrees of freedom in this case reduces to a quantum theory of the massless scalar field  $\psi$  on a fictitious background [4]. There is a convenient way of writing the solutions of the Klein-Gordon equation,

$$\psi(\theta, T) = \sum_{m \in \mathbf{Z}} f_m(\theta, T) A_m + \overline{f_m(\theta, T)} \overline{A_m}, \quad (14)$$

where the bar denotes complex conjugation, the  $A_m$ 's are arbitrary constants and  $f_0(\theta, T) = (1/2)(\ln T - i)$ ,  $f_m(\theta, T) = (1/2)H_0^{(1)}(|m|T)e^{im\theta}$  for  $m \neq 0$ , with  $H_0^{(1)}$  the 0th-order Hankel function of the first kind.

For the quantization of this system one can use the Fock procedure, starting from the one-particle Hilbert space  $\mathcal{H}_0$  for which an appropriate complex structure  $J_V$  is needed that must be compatible with the symplectic structure  $\Omega_V$ . It can be shown [4] that in this case the complex structure can be chosen as,

$$J_V(a \ln T) := -a \quad \text{and} \quad J_V(a) := a \ln T, \quad (15)$$

for  $m = 0$  and as,

$$J_V(J_0(|m|T)) := N_0(|m|T) \quad \text{and} \quad J_V(N_0(|m|T)) := -J_0(|m|T), \quad (16)$$

for  $m \neq 0$ , where  $a$  is a constant. The (fiducial) Hilbert space can be represented as  $\mathcal{F} = \mathcal{H}_g \otimes \bar{\mathcal{F}}$ , where  $\mathcal{H}_g$  is the Hilbert space in which the operators  $\hat{q}$  and  $\hat{p}$  are well defined, and  $\bar{\mathcal{F}}$  is the symmetric Fock space (associated to the “one-particle” Hilbert space  $\mathcal{H}_0$ ) in which the field operator  $\hat{\psi}$  can be written as,

$$\hat{\psi}(\theta, T) = \sum_{m \in \mathbf{Z}} f_m(\theta, T) \hat{A}_m + \overline{f_m(\theta, T)} \hat{A}_m^\dagger, \quad (17)$$

in terms of the creation and annihilation operators. The space of physical states,  $\mathcal{F}_p$ , is the subspace of  $\mathcal{F}$  defined by

$$: \hat{P}_\theta : |\Psi\rangle_p = 2 \sum_{m \in \mathbf{Z}} m \hat{A}_m^\dagger \hat{A}_m |\Psi\rangle_p = 0. \quad (18)$$

Finally, the Hamilton operator can be expressed as,

$$\hat{H} = \frac{\pi}{2T} : \hat{p}(\hat{A}_0 + \hat{A}_0^\dagger)^2 : + \sum_{n \in \mathbf{Z}} \hat{p}(\alpha_n \hat{A}_n \hat{A}_{-n} + \overline{\alpha_n} \hat{A}_n^\dagger \hat{A}_{-n}^\dagger + 2\beta_n \hat{A}_n^\dagger \hat{A}_n) \quad (19)$$

where  $\alpha_n(T) = (T/4)n^2[(H_0^{(1)})^2 + (H_1^{(1)})^2]$  and  $\beta_n(T) = (T/4)n^2[H_1^{(1)}H_1^{(2)} + H_0^{(1)}H_0^{(2)}]$  for  $n \neq 0$ . Although this Hamiltonian leaves  $\mathcal{F}_p$  invariant, it has the disadvantage that the

vacuum state is not an eigenvector of it with zero eigenvalue. In order to avoid this difficulty, a new set of creation and annihilation operators has been recently proposed [5]:

$$\hat{a}_0 = \frac{i}{\sqrt{2}}(\sqrt{3}\hat{A}_0 + \hat{A}_0^\dagger), \quad (20)$$

and

$$\hat{a}_n = \sqrt{\frac{\pi}{2}}(\tilde{\alpha}_n\hat{A}_n + \tilde{\beta}_n\hat{A}_{-n}^\dagger), \quad (21)$$

where  $\tilde{\alpha}_n = \alpha_n/(|n|\tilde{\beta}_n)$  and  $\tilde{\beta}_n = \sqrt{(\beta_n - |n|/\pi)/|n|}$ . It should be noted that our choice differs from that in [5], which does not satisfy the relations  $[\hat{a}_n, \hat{a}_{n'}^\dagger] = \delta_{n,n'}$ . In this new set the Hamilton operator can be written as,

$$:\hat{H} := \frac{1}{T}\hat{p}\hat{P}_0^2 + \frac{2}{\pi} \sum_{n \in \mathbf{Z}} |n| \hat{p} \hat{a}_n^\dagger \hat{a}_n, \quad (22)$$

with  $\hat{P}_0 = i\sqrt{\pi}/(1 + \sqrt{3})(\hat{a}_0^\dagger - \hat{a}_0)$ . For this Hamiltonian, the vacuum state is in fact an eigenvector with zero eigenvalue. The question arises whether locally the two different sets of creation and annihilation operators lead to different quantizations. If so, one would need to investigate the problem of unitary implementability (to be treated in the next section) for both sets separately. To answer this question we have analyzed the coefficients that relate the old operators  $\hat{A}_n$  with the new ones  $\hat{a}_n$ . A straightforward calculation shows that they satisfy the relationships,

$$\sum_k (\tilde{\alpha}_{mk} \tilde{\alpha}_{nk} - \tilde{\beta}_{mk} \tilde{\beta}_{nk}) = \delta_{m,n}, \quad (23)$$

and

$$\sum_k (\tilde{\alpha}_{mk} \tilde{\beta}_{nk} - \tilde{\beta}_{mk} \tilde{\alpha}_{nk}) = 0, \quad (24)$$

where  $\tilde{\alpha}_{mk} = \tilde{\alpha}_m \delta_{m,k}$  and  $\tilde{\beta}_{mk} = \tilde{\beta}_m \delta_{-m,k}$ . Moreover, the coefficients  $\tilde{\beta}_{mk}$  are square-summable. To see this, let  $C(|m|T) = (\beta_m - |m|/\pi)/|m|$  and let  $N$  be a positive integer such that  $NT \gg 1$ , then

$$\sum_{m,n} |\tilde{\beta}_{nm}|^2 = \sum_m |\tilde{\beta}_m|^2 = 2 \sum_{m=1}^{N-1} C(mT) + 2 \sum_{m=N}^{\infty} C(mT) \quad (25)$$

and the condition that the coefficients  $\tilde{\beta}_{mk}$  are square-summable is equivalent to

$$\sum_{m=N}^{\infty} C(mT) < \infty \quad (26)$$

Now, from the expansions for  $J_n(x)$  and  $N_n(x)$  [19]; i.e.,

$$\begin{aligned} J_n(x) &= \sqrt{\frac{2}{\pi x}} \left[ P_n(x) \cos \left( x - \frac{(n + \frac{1}{2})\pi}{2} \right) - Q_n(x) \sin \left( x - \frac{(n + \frac{1}{2})\pi}{2} \right) \right], \\ N_n(x) &= \sqrt{\frac{2}{\pi x}} \left[ P_n(x) \sin \left( x - \frac{(n + \frac{1}{2})\pi}{2} \right) + Q_n(x) \cos \left( x - \frac{(n + \frac{1}{2})\pi}{2} \right) \right], \end{aligned} \quad (27)$$

where

$$P_n(x) = 1 - \frac{(4n^2 - 1)(4n^2 - 9)}{2!(8x)^2} + \frac{(4n^2 - 1)(4n^2 - 9)(4n^2 - 25)(4n^2 - 49)}{4!(8x)^4} - \dots, \quad (28)$$

$$Q_n(x) = \frac{(4n^2 - 1)}{1!(8x)} - \frac{(4n^2 - 1)(4n^2 - 9)(4n^2 - 25)}{3!(8x)^3} + \dots,$$

it is easy to see that

$$C(kT) = \frac{Tk}{4} \left[ \frac{2}{\pi Tk} (P_1^2(kT) + P_0^2(kT) + Q_1^2(kT) + Q_0^2(kT)) \right] - \frac{1}{\pi}. \quad (29)$$

That is, for  $kT \gg 1$  we obtain that

$$C(kT) = \frac{Tk}{4} \left[ \frac{2}{\pi Tk} (2 + O(1/(kT)^2)) \right] - \frac{1}{\pi} \quad (30)$$

where  $O(1/(kT)^2)$  contains all the terms of the form  $\frac{c_i}{(kT)^n}$ , with  $c_i$  some (real) constants and  $\mathbf{N} \ni n \geq 2$ . Because  $\sum_k \frac{c_i}{(kT)^n}$  converges (the Riemann zeta function defined by  $\zeta(p) = \sum_{k=1}^{\infty} k^{-p}$  is divergent for  $p \leq 1$  and convergent for  $p > 1$ ) then Eq. (30) implies Eq. (26) and therefore the coefficients  $\tilde{\beta}_{mk}$  are square-summable. Thus, the transformation between the two different sets of creation and annihilation operators is a unitary Bogoliubov transformation [15,17]. Consequently, the second set of operators,  $\hat{a}_n$ , corresponds only to the choice of a second complete set of modes  $\tilde{f}_n$ . Thus, we can equivalently choose any of these two sets of creation and annihilation operators given above. We will perform the analysis of the quantum implementability of the classical dynamical evolution in Sec. III, using the operators  $\hat{A}_n$  and the complex structure given above.

### III. FUNCTIONAL EVOLUTION

It is known that for a massless, free, real scalar field propagating on a  $(n+1)$ -dimensional static spacetime with topology  $T^n \times \mathbf{R}$ , dynamical evolution along arbitrary spacelike foliations is unitarily implemented on the same Fock space as that associated with inertial foliations if  $n = 1$  [13] and will not be, in general, unitarily implemented if  $n > 1$  [14]. However, for the (special) case in which we consider time evolution of the Klein-Gordon field between any two flat spacelike Cauchy surfaces, dynamical evolution is unitarily implementable for all positive integers  $n$ . In this section we will see that this result actually does not extend to our case, where the spatial slices have the same topology (tori) but now they are expanding.

#### A. The symplectic transformation of classical dynamics

It is generally known that the phase space of a real, linear Klein-Gordon field propagating on a globally hyperbolic background spacetime  $(M \simeq \Sigma \times \mathbf{R}, g_{ab})$  with  $\Sigma$  a compact spacelike Cauchy surface, can be alternatively described by the space  $\Gamma$  of Cauchy data, that is



$\{(\varphi, \pi) | \varphi : \Sigma \rightarrow \mathbf{R}, \pi : \Sigma \rightarrow \mathbf{R}; \varphi, \pi \in C^\infty(\Sigma)\}$ , or by the space  $V$  of smooth solutions to the Klein-Gordon equation which arises from initial data on  $\Gamma$  [16]. Given an embedding  $E$  of  $\Sigma$  as a Cauchy surface  $E(\Sigma)$  in  $M$ , there is a natural isomorphism  $I_E : \Gamma \rightarrow V$ , obtained by taking a point in  $\Gamma$  and evolving from the Cauchy surface  $E(\Sigma)$  to get a solution of  $(g^{ab}\nabla_a\nabla_b - m^2)\psi = 0$ . That is, the specification of a point in  $\Gamma$  is appropriate initial data for determining a solution to the equation of motion. The inverse map,  $I_E^{-1} : V \rightarrow \Gamma$ , takes a point  $\psi \in V$  and finds the Cauchy data induced on  $\Sigma$  by virtue of the embedding  $E$ :  $\varphi = E^*\psi$  and  $\pi = E^*(\sqrt{h}\mathcal{L}_n\psi)$ , where  $\mathcal{L}_n$  is the Lie derivative along the normal to the Cauchy surface  $E(\Sigma)$  and  $h$  is the determinant of the induced metric on  $E(\Sigma)$ .

It is worth pointing out that  $\Gamma$  and  $V$  are equipped with a (natural) symplectic structure  $\Omega_\Gamma$  and  $\Omega_V$ , respectively, that provides the space of classical observables, which are (alternatively) represented by smooth real valued functions on  $\Gamma$  or  $V$ , with an algebraic structure via the Poisson bracket. On the space of solutions  $V$ , the symplectic structure is

$$\Omega_V(\psi_1, \psi_2) = \int_{E(\Sigma)} \sqrt{h}(\psi_2 \mathcal{L}_n \psi_1 - \psi_1 \mathcal{L}_n \psi_2) \quad (31)$$

while on the space of Cauchy data,  $\Gamma$ , is given by

$$\Omega_\Gamma((\varphi_1, \pi_1), (\varphi_2, \pi_2)) = \int_\Sigma (\varphi_2 \pi_1 - \varphi_1 \pi_2) \quad (32)$$

From Eqs(31)-(32) and the specification of the isomorphism  $I_E$ , it is obvious that  $\Omega_\Gamma = I_E^* \Omega_V$ ; i.e.,  $I_E$  is a symplectic map.

Now, let  $E_I(\Sigma)$  and  $E_F(\Sigma)$  be any given initial and final Cauchy surfaces, represented by embeddings  $E_I$  and  $E_F$ . *Time evolution* from  $E_I(\Sigma)$  to  $E_F(\Sigma)$  can be viewed as a bijection  $t_{(E_I, E_F)} : \Gamma \rightarrow \Gamma$  on the space of Cauchy data [14]:  $t_{(E_I, E_F)} := I_{E_F}^{-1} \circ I_{E_I}$ . Thus, the recipe is: (a) take initial data on  $E_I(\Sigma)$ , (b) evolve it to a solution of the Klein-Gordon equation, and (c) find the corresponding pair induced on  $E_F(\Sigma)$  by this solution. Notice that this map also defines *time evolution* on the space of solutions,  $V$ , through the natural induced bijection  $T_{(E_I, E_F)} := I_{E_I} \circ t_{(E_I, E_F)} \circ I_{E_F}^{-1}$ . The three steps of the recipe are now: (a) take a solution to the field equation, (b) find the data induced on  $E_F(\Sigma)$ , and (c) take the data as initial data on  $E_I(\Sigma)$  and find the resulting solution. It is straightforward to see, from the embedding independence of (31) and from (32), that each transformation is a symplectic isomorphism. i.e.,  $t_{(E_I, E_F)}^* \Omega_\Gamma = \Omega_\Gamma$  and  $T_{(E_I, E_F)}^* \Omega_V = \Omega_V$ .

For our particular case, we shall construct dynamical evolution between any two flat Cauchy surfaces  $E_I(T^2) := (T_I, x^i)$  and  $E_F(T^2) := (T_F, x^i)$ , where  $x^i = (\theta, \sigma) \in (0, 2\pi)$  are coordinates on  $T^2$  and  $T$  is the smooth “time coordinate” on  $(M \simeq T^2 \times \mathbf{R}, {}^{(t)}g_{ab})$  such that each surface of constant  $T$  is a Cauchy surface. Let us denote by  $\tilde{\psi}$  the resulting solution from the action of  $T_{(E_I, E_F)}$  on  $\psi$ . Following the prescription, we first have to find the induced data on  $E_F(T^2)$ :

In general  $\varphi_F = E_F^*\psi$  and  $\pi_F = E_F^*(\sqrt{h_F}\mathcal{L}_{n_F}\psi)$ , since in our case  $E_F(T^2) = (T_F, x^i)$  and  $\psi$  depend on the coordinates  $T$  and  $\theta$  only, we thus have that  $\varphi_F = \psi(\theta, T_F)$  and  $\pi_F = [T\partial_T\psi(\theta, T)]|_{T=T_F}$ . Thus, from the explicit form (14) for solutions of the Klein-Gordon equation, we have that

$$\varphi_F = \Im(A_0) + \Re(A_0) \ln T_F + \sum_{m \neq 0} \Re[B_m H_0^{(1)}(|m|T_F)] \quad (33)$$

$$\pi_F = \Re(A_0) - T_F \sum_{m \neq 0} |m| \Re[B_m H_1^{(1)}(|m|T_F)] \quad (34)$$

where  $B_m = A_m e^{im\theta}$ .

The next step in the prescription is to take the pair  $(\varphi_F, \pi_F)$  as initial data on  $E_I(T^2)$  and find the resulting solution  $\tilde{\psi}$ . That is, we have to solve for  $\{\tilde{A}_k, \overline{\tilde{A}_k}\}_{k \in \mathbf{Z}}$  the following system

$$\psi(\theta, T_F) = \tilde{\psi}(\theta, T_I) \quad (35)$$

$$[T \partial_T \psi(\theta, T)]|_{T=T_F} = [T \partial_T \tilde{\psi}(\theta, T)]|_{T=T_I} \quad (36)$$

where  $\tilde{\psi}(\theta, T_I) = E_I^* \tilde{\psi}$  and  $[T \partial_T \tilde{\psi}(\theta, T)]|_{T=T_I} = E_I^*(\sqrt{h_I} \mathcal{L}_{n_I} \tilde{\psi})$  are explicitly given by

$$\tilde{\psi}(\theta, T_I) = \Im(\tilde{A}_0) + \Re(\tilde{A}_0) \ln T_I + \sum_{m \neq 0} \Re[\tilde{B}_m H_0^{(1)}(|m|T_I)] \quad (37)$$

$$[T \partial_T \tilde{\psi}(\theta, T)]|_{T=T_I} = \Re(\tilde{A}_0) - T_I \sum_{m \neq 0} |m| \Re[\tilde{B}_m H_1^{(1)}(|m|T_I)] \quad (38)$$

with  $\tilde{B}_m = \tilde{A}_m e^{im\theta}$ .

Using the orthogonality property  $\oint e^{i(n-m)\theta} = 2\pi \delta_{n,m}$ , the well-known relation  $H_0^{(1)}(x) H_1^{(1)}(x) - H_1^{(1)}(x) H_0^{(1)}(x) = \frac{4i}{\pi x}$  (where  $x > 0$ ) and the explicit expression for the fields, given by equations (33), (34), (37) and (38), it is not difficult to see that the system (35)-(36) is solved by

$$\Re(\tilde{A}_0) = \Re(A_0) \quad (39)$$

$$\Im(\tilde{A}_0) = \Im(A_0) + \Re(A_0) \ln(T_F/T_I) \quad (40)$$

for  $k = 0$ , and

$$\tilde{A}_k = \frac{i\pi}{4} [F(y_k, x_k) - \overline{F(x_k, y_k)}] A_k + \frac{i\pi}{4} [G(y_k, x_k) - G(x_k, y_k)] \overline{A_{-k}} \quad (41)$$

for all  $k \neq 0$ , where  $x_k := |k|T_I$ ,  $y_k := |k|T_F$ ,  $F(r, s) := r H_1^{(1)}(r) \overline{H_0^{(1)}(s)}$  and  $G(r, s) = r \overline{H_1^{(1)}(r)} H_0^{(1)}(s)$ .

Therefore, the symplectic transformation  $T_{(E_I, E_F)}$  defines, and is defined by, a transformation of  $\overline{A_m}$ :

$$\overline{\tilde{A}_k} = \sum_{l \in \mathbf{Z}} \chi_{kl} A_l + \xi_{kl} \overline{A_l} \quad (42)$$

where

$$\chi_{k0} = -\frac{i}{2} \ln(T_F/T_I) \delta_{k,0}, \quad \chi_{kl} = \frac{i\pi}{4} [\overline{G(x_l, y_l)} - \overline{G(y_l, x_l)}] \delta_{l,-k} \quad (\forall l \neq 0) \quad (43)$$

$$\xi_{k0} = [1 - \frac{i}{2} \ln(T_F/T_I)] \delta_{k,0}, \quad \xi_{kl} = \frac{i\pi}{4} [F(x_l, y_l) - \overline{F(y_l, x_l)}] \delta_{l,k} \quad (\forall l \neq 0) \quad (44)$$

Obviously  $\overline{\tilde{A}_k} = \overline{A_k}$  for all  $k \in \mathbf{Z}$  when  $T_I = T_F$  (i.e., when  $T_{(E_I, E_F)}$  is the identity map).

## B. Quantum Implementability

The question we want to address in this part is whether or not classical evolution on the fictitious background is implementable at the quantum level. A particularly convenient approach to this issue is given by the algebraic approach of QFT, since the notion of implementability of symplectic transformations on a Hilbert space formulation is defined in a natural way [14]. The main idea in the algebraic approach is to formulate the quantum theory in such a way that the observables become the relevant objects and the quantum states are “secondary”, they are taken to “act” on operators to produce numbers. The basic ingredients of this formulation are two, namely : (1) a  $C^*$ -algebra  $\mathcal{A}$  of observables, and (2) states  $\omega : \mathcal{A} \rightarrow \mathbf{C}$ , which are positive linear functionals ( $\omega(A^*A) \geq 0 \ \forall A \in \mathcal{A}$ ) such that  $\omega(\mathbf{1}) = 1$ . The value of the state  $\omega$  acting on the observable  $A$  can be interpreted as the expectation value of the operator  $A$  on the state  $\omega$ , i.e.,  $\langle A \rangle = \omega(A)$ .

For free (linear) fields it is possible to construct the Weyl algebra of quantum abstract operators from the elementary classical observables (equipped with an algebraic structure given by the Poisson bracket). The elements of this  $C^*$ -algebra are taken to be the fundamental observables for the quantum theory, thus the (natural) algebra  $\mathcal{A}$  for free fields is the Weyl algebra. Let  $(Y, \Omega_Y)$  be a symplectic vector space, each generator  $W(y)$  of the Weyl algebra is the “exponentiated” version of the linear observable  $\Omega_Y(y, \cdot)$ . These generators satisfy the Weyl relations<sup>1</sup>:

$$W(y)^* = W(-y), \quad W(y_1)W(y_2) = e^{-\frac{i}{2}\Omega_Y(y_1, y_2)}W(y_1 + y_2) \quad (45)$$

Given a symplectic transformation  $f$  on  $Y$ , there is an associated  $*$ -automorphism of  $\mathcal{A}$ ,  $\alpha_f : \mathcal{A} \rightarrow \mathcal{A}$ , defined by  $\alpha_f \cdot W(y) := W(f[y])$ . In particular, the symplectic transformation  $T_{(E_I, E_F)}$  representing time evolution from  $E_I = (T_I, x^i)$  to  $E_F = (T_F, x^i)$  defines the  $*$ -automorphism  $\alpha_{(E_I, E_F)}$ . Thus, from the algebraic point of view, if we assign the state  $\omega$  to the initial time as represented by the embedding  $E_I$ , the expectation value of the observable  $W \in \mathcal{A}$  on  $E_I$  is given by

$$\langle W \rangle_{E_I} = \omega(W) \quad (46)$$

Let us consider the Weyl generator  $W(\psi)$  labeled by  $\psi$ . Under the symplectic transformation  $T_{(E_I, E_F)}$ , the label goes to  $\tilde{\psi}$  and the relation between  $W(\psi)$  and  $W(\tilde{\psi})$  is given by  $\alpha_{(E_I, E_F)}W(\psi) = W(\tilde{\psi})$ . Since  $T_{(E_I, E_F)}$  dictates time evolution at classical level, one can interpret the change  $W(\psi) \rightarrow \alpha_{(E_I, E_F)}W(\psi)$  as a counterpart in the observables. That is,  $W(\psi) \rightarrow \alpha_{(E_I, E_F)}W(\psi)$  is the mathematical representation of time evolution of observables in the Heisenberg picture. Thus, while in the Heisenberg picture the expectation value of the observable  $W(\psi)$  at final time is given by  $\langle W(\psi) \rangle_{E_F} = \omega(\alpha_{(E_I, E_F)} \cdot W(\psi))$ , in the Schrödinger picture is  $\langle W(\psi) \rangle_{E_F} = \omega_{E_F}(W(\psi))$ . Therefore, the final state  $\omega_{E_F}$ , obtained by evolving the initial state  $\omega$ , is given by  $\omega \circ \alpha_{(E_I, E_F)}$ .

Now, in order to know if time evolution between any two flat Cauchy surfaces is well defined in the framework of the Hilbert space formulation, we have to introduce the GNS

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<sup>1</sup>The CCR that correspond to operators  $\hat{\Omega}_Y(y, \cdot)$  get now replaced by the quantum Weyl relations.

construction that tells us how the quantization in the old sense (that is, a representation of the Weyl relations on a Hilbert space) and the algebraic approach are related:

*Let  $\mathcal{A}$  be a  $C^*$ -algebra with unit and let  $\omega : \mathcal{A} \rightarrow \mathbb{C}$  be a state. Then there exist a Hilbert space  $\mathcal{H}$ , a representation  $\pi : \mathcal{A} \rightarrow L(\mathcal{H})$  and a vector  $|\Psi_0\rangle \in \mathcal{H}$  such that,  $\omega(A) = \langle \Psi_0, \pi(A)\Psi_0 \rangle_{\mathcal{H}}$ . Furthermore, the vector  $|\Psi_0\rangle$  is cyclic. The triplet  $(\mathcal{H}, \pi, |\Psi_0\rangle)$  with these properties is unique (up to unitary equivalence).*

With this in hand, we have a precise way to ‘go down’ transformations on the  $C^*$ -algebra to a given Hilbert space representation. Thus, a symplectic transformation  $f : Y \rightarrow Y$ , with corresponding algebra automorphism  $\alpha_f : \mathcal{A} \rightarrow \mathcal{A}$ , is unitarily implementable [14] if there is a unitary transformation  $U : \mathcal{H} \rightarrow \mathcal{H}$  on the Hilbert space  $\mathcal{H}$  such that, for any  $W \in \mathcal{A}$ ,  $U^{-1}\pi(W)U = \pi(\alpha_f \cdot W)$ .

Because  $\omega$  and its transform  $\omega \circ \alpha_f$  will not always define unitarily equivalent Hilbert space representations, thus not all symplectic transformations  $f$  will be implementable in field theory. In our case, we are interested on the implementability of  $f = T_{(E_I, E_F)}$  on the symmetric Fock space  $\bar{\mathcal{F}}$ , constructed from the so-called “one-particle” Hilbert space,  $\mathcal{H}_0$ , which elements are the complex functions

$$\Psi = \sum_{m \in \mathbf{Z}} \overline{f_m(\theta, T)} \overline{A_m} \quad (47)$$

determined by the natural splitting of  $V_{\mathbf{C}}$ , the complexification of  $V$ , on negative and *positive* parts through the complex structure  $J_V$ .

The continuous<sup>2</sup> transformation (42) defines a pair of bounded linear maps  $\xi : \mathcal{H}_0 \rightarrow \mathcal{H}_0$  and  $\chi : \mathcal{H}_0 \rightarrow \overline{\mathcal{H}_0}$ , where  $\overline{\mathcal{H}_0}$  is the complex conjugate space to  $\mathcal{H}_0$ . With  $\Psi$  given by (47), we have

$$\xi \cdot \Psi = \sum_{m, l \in \mathbf{Z}} \overline{f_m(\theta, T)} \xi_{ml} \overline{A_l} \quad (48)$$

and

$$\chi \cdot \Psi = \sum_{m, l \in \mathbf{Z}} f_m(\theta, T) \overline{\chi_{ml}} \overline{A_l} \quad (49)$$

The automorphism  $\alpha_{(E_I, E_F)}$  associated with  $T_{(E_I, E_F)}$  is unitarily implementable with respect to the Fock representation  $(\bar{\mathcal{F}} = \mathcal{F}_s(\mathcal{H}_0), \pi)$  if and only if the operator  $\chi$  is Hilbert-Schmidt [17]. i.e., iff

$$\sum_{m, l \in \mathbf{Z}} |\chi_{lm}|^2 < \infty \quad (50)$$

Since  $\sum_{m, l \in \mathbf{Z}} |\chi_{lm}|^2 = \frac{1}{2} \ln(T_F/T_I)^2 + \sum_{l, m \neq 0} |\chi_{lm}|^2$ , then according to Eq.(43) condition (50) is equivalent to

$$\sum_{m \neq 0} (\Re[g_m(x_m, y_m)])^2 < \infty \quad \text{and} \quad \sum_{m \neq 0} (\Im[g_m(x_m, y_m)])^2 < \infty \quad (51)$$

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<sup>2</sup>Actually, it can be shown that there is a constant  $b$  such that, for all  $\psi \in V$ ,  $\|T_{(E_I, E_F)}\psi\| \leq b\|\psi\|$  in the norm  $\|\psi\|^2 = \Omega_V(J_V\psi, \psi)$ .

where  $g_m(r, s) := \overline{G(r, s)} - \overline{G(s, r)}$ . Using the definition of Hankel function in terms of Bessel and Neumann functions (for  $n = 0$  or  $1$ ), the first condition in (51) can be written as follows

$$\sum_{m=1}^{\infty} (\Lambda_m[a, y_m])^2 < \infty \quad (52)$$

where  $\Lambda_m[a, y_m] := m[a(J_0(y_m)J_1(ay_m) - N_0(y_m)N_1(ay_m)) + N_1(y_m)N_0(ay_m) - J_1(y_m)J_0(ay_m)]$  and  $a := T_I/T_F$ . In the asymptotic region  $x \gg 1$  (for  $n = 0$  or  $1$ ) the behavior of Bessel and Neumann functions is given by  $J_n(x) \approx \sqrt{\frac{2}{\pi x}} \cos(x - (n + \frac{1}{2})\frac{\pi}{2})$  and  $N_n(x) \approx \sqrt{\frac{2}{\pi x}} \sin(x - (n + \frac{1}{2})\frac{\pi}{2})$  respectively, then  $\Lambda_m[a, y_m] \approx \frac{2m}{\sqrt{a\pi y_m}}(a - 1) \cos(y_m(1 + a) - \pi)$  for  $m \gg 1$  is the asymptotic behavior of  $\Lambda_m[a, y_m]$  and (52) becomes

$$\frac{4(1 - a)^2}{a\pi^2 T_F^2} \sum_{m=N}^{\infty} \cos^2(T_F(1 + a)m) < \infty \quad (53)$$

where  $N \gg 1$ . Notice that for the case  $a = 1$  this condition is trivially satisfied, as expected.

Thus, unitary implementability implies that (53) is satisfied for all  $a \in (0, 1)$  and hence, if there are particular values of  $T_F$  ( $r_0 > 0$ ) and  $a$  ( $a_0 \in (0, 1)$ ) such that  $\sum_{m=N}^{\infty} \cos^2(r_0(1 + a_0)m)$  diverges, then  $\alpha_{(E_I, E_F)}$  will not be unitarily implemented. In particular, by choosing  $T_F = \frac{\pi}{1+a}$  every integer  $m \geq N$  corresponds to a maximum of  $\cos^2(T_F(1 + a)x)$  and therefore the sum in (53) diverges. Thus, the transformation associated with  $T_{(E_I, E_F)}$  is *not* unitarily implementable with respect to the Fock representation  $(\mathcal{F}_s(\mathcal{H}_0), \pi)$  and hence classical time evolution, dictated by  $T_{(E_I, E_F)}$ , does not have a quantum analog in the Hilbert space formulation via a unitary operator. In this case, the Schrödinger picture is not available to describe functional evolution using the Fock space representation of the quantum theory, consequently the “Schrödinger equation” associated with the Hamilton operator  $\hat{H}(T)$  can not be interpreted as an evolution equation (on the fictitious background) for quantum states.

#### IV. DISCUSSION AND CONCLUSIONS

In this work we have analyzed the quantization of polarized Gowdy  $T^3$  cosmological models as carried out by Pierri. We have found explicitly the symplectic transformation that determines the classical dynamical evolution  $T_{(E_I, E_F)}$  given by the phase space function  $T$ . We have shown that this symplectic transformation does not have a quantum analog in the Hilbert space formulation via a unitary operator. This means that the classical dynamics of Gowdy  $T^3$  cosmological models cannot be implemented in the context of Pierri’s quantization procedure. Let us now discuss the implications of this negative result for two related areas, namely for canonical quantum gravity and for quantization of fields on curved manifolds.

##### *Canonical Quantum Gravity.*

First of all let us recall that in canonical quantum gravity, the theory is defined over an “abstract” manifold  $\Sigma$  which in our case is given by  $\Sigma = T^2$ . There is no spacetime and therefore no notion of an embedding of  $\Sigma$  into this spacetime. What we have is, in the gauge fixed scenario, a reduced phase space representing the true degrees of freedom, and,

in the quantum theory, a Hilbert space of physical states and physical observables defined on it. This is the frozen formalism description. When the deparametrization procedure was introduced classically, an artificial notion of time evolution was created that allows to “evolve” any set of (physical) initial conditions on  $\Sigma$  into a one parameter family of initial conditions with a precise spacetime interpretation. This one parameter family is generated via a canonical transformation generated by the reduced Hamiltonian. In quantum gravity, however, the parameter  $T$  can not be thought, a priori, as a time function in a spacetime for the only reason that a spacetime notion is absent. In which sense is then useful the parameter  $T$ ? Recall from the discussion in Sec. III that the notion of time evolution in the algebraic formulation is well defined, giving rise to the Heisenberg picture: We have a unique state  $|\Psi\rangle_{T_0}$ , defined on a preferred and fixed  $\Sigma_0$ , and operators acting on it which could be “time dependent”. This is the place where the parameter  $T$  plays a central role. We can have, for instance, a one parameter family of observables, say  $\hat{V}_T$ , corresponding to “the volume of the Universe at time  $T$ ” [5]. In the standard formulation of quantum theory, where a unitary evolution operator  $\hat{U}(T, T_0)$  exists, we can relate the operators belonging to the family via unitary transformations. One can also construct the Schrödinger picture and have a one-parameter family of states that “evolve” in time  $T$ , using the standard construction. Canonical quantum gravity is most naturally constructed in the Heisenberg picture, where the physical operators correspond to the so called “evolving constants of the motion”. Therefore, any operator  $\mathcal{O}_T$  labeled by the time  $T$ , if it is well defined (i.e. if it leaves the Hilbert space, or a dense subset of it, invariant), can be regarded as a physically meaningful object representing the classical observable at “time  $T$ ”. What is lost in the absence of the operator  $\hat{U}(T, T_0)$ , as is our case, is the “unitary equivalence” of the operators  $\mathcal{O}_T$  for all values of  $T$ . In this sense, unitary time evolution and the Schrödinger picture are lost. Unitary evolution is one of the pillars of present quantum theory, and theories that do not satisfy this property suffer from the rejection of the community, since the theory becomes unable to make predictions due to the lack of conservation of probability. This would be the case, for instance, in the event of the evaporation of a black hole via Hawking radiation. Physicists have always tried to avoid such descriptions and look for explanations that are “unitary”. However, as we would like to argue, canonical quantum gravity is conceptually very different from the standard description of quantum theory with a preferred and external Newtonian time, so one should look for more involved arguments before dismissing a particular theory. In the Hamiltonian description, time evolution is pure gauge, so strictly speaking one should only meaningfully discuss physical observables on the reduced phase space. There is no time evolution and no dynamics. Any deparametrization is classically equivalent, giving rise to fictitious dynamics via canonical transformations. There is no compelling reason to expect that there is a preferred deparametrization that will be meaningful quantum-mechanically. Thus, there is no logical contradiction to the result that a particular choice can not be unitarily implemented.

Within this perspective, the parameter  $T$  that was introduced artificially at the classical level bears no fundamental physical significance. The fact that the quantum theory does not endorse this choice should not be enough reason to dismiss it. However, it should be clear that the absence of unitary transformation reduces significantly the importance of this quantization, since the Heisenberg operators  $\mathcal{O}_T$  are not well defined. That is, their spectra, expectation values, etc., depend on the choice of the value of  $T_0$ . Had we chosen a

different value  $T = T'_0$  and therefore different Heisenberg state  $|\Psi\rangle_{T'_0}$ , we would get different operators  $\mathcal{O}'_T \neq \mathcal{O}_T$  (for the same value of  $T$ ). A minimum requirement for the consistency of the quantization is that the operators  $\mathcal{O}'_T$  and  $\mathcal{O}_T$  be (unitary) equivalent. Thus, the quantization is physically unacceptable.

However, it is not completely clear whether this negative result holds for any choice of a set of creation and annihilation operators (or, equivalently, choice of complex structure  $J$ ). The original choice in Ref. [4] seems natural from the viewpoint of the explicit form of the solutions of the Klein-Gordon equation, and the fact that is time independent and therefore there is no “particle creation”. However, further work is needed in order to understand whether there exist different choices of  $J$  and therefore of representations of the CCR for which “time evolution” is a well defined concept. Unitary implementability might even be a criteria leading to a physically relevant quantization.

#### *Quantum Fields on Curved Surfaces.*

The issue of formulating time evolution between arbitrary Cauchy surfaces in the quantum theory of fields goes back to the work of Dirac [18]. However, it is only recently that unitary implementability of arbitrary time evolution has been considered. Somewhat surprisingly, it has been recognized that even for free fields on Minkowski spacetime, time evolution between arbitrary Cauchy surfaces is not unitarily implementable in three and higher space-time dimensions [14]. The failure is in general attributed to the fact that time evolution between arbitrary surfaces is not generated by an isometry of the background metric [12,14]. It is also known that in two dimensions, for the standard quantization coming from the symmetries of the system, time evolution is well defined for arbitrary Cauchy surfaces with topology of a circle. However, for our model, even when it is a truly two dimensional model ( $\psi$  depends only on  $\theta$  and  $T$ ), it does not satisfy a free scalar equation (it is instead related to a Liouville model). Therefore, there is no contradiction with the fact that time evolution is not unitary. From the three dimensional perspective, the theory is given by a free scalar field on a flat background, but in which the vector field  $\partial/\partial T$  that generates the natural time evolution is not an isometry of the background spacetime. Thus, it is interesting to see that in this case, even for the simplest Cauchy surfaces (flat and parallel in the given chart), time evolution is not implementable. It is also interesting to note that particle creation and non-unitary time evolution do not imply each other, as noted in [14]. As previously mentioned, it is not clear whether different representations of the CCR would yield unitary quantum theories. Namely, is there a choice of  $J$  that will render the theory unitary? Would it be unique? We shall leave these questions for future investigations.

Note added: After submitting this paper, we learned that similar results were independently found by Torre [20].

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